16. STACEY T.R. and JONGH C.L., Stress fracturing around a deep level bored tunnel, J. South Afr., Inst. Mining and Metallurg., 78, 5, 1977.
17. NIKITIN L.V. and ODINTSEV V.N., Formation of extended, closed separation cracks in brittle rocks, Dokl. Akad. Nauk SSSR, 293, 4, 1987.

Translated by M.D.F.

J. Appl. Maths Mechs, Vol. 55, No. 1, pp. 93-99, 1991
Printed in Great Britain
0021-8928/91 $15.00+0.00

DUAL FORMULATIONS OF THE BOUNDARY-ELEMENTS METHOD. APPLICATION TO ELASTICITY THEORY PROBLEMS FOR INHOMOGENEOUS BODIES*

V.YA. TERESHCHENKO

Alternative variational formulations are considered for the boundary-elements method (BEM) that utilize the formulation of minimization problem of boundary functionals and generalized Trefftz functions of linear elasticity theory /1/. The variational solutions are approximated by using boundary potentials with the desired density: the formulation in displacements (line) in place of the interpolation considered earlier of the double layer potential (DLP) density uses interpolation on the boundary element (BE) of the simple layer potential (SLP) density according to the nodal values of the displacements; the dual formulation is interpolation on the BE of the PLP density according to the nodal values of the stresses.

It is best to use the formulation for solving problems of elasticity theory with mixed boundary conditions, contact problems. In particular, the dual formulation turns out to be effective in solving problems for elastic media with discontinuous elasticity coefficients (piecewise-homogeneous); adjoint conditions must be realized in the corresponding variational problem for both the displacement vector and for the stress vector on the surface of discontinuity of the coefficients. The results obtained in /1/ and in this paper are compared with the results arising from other BEM formulations.

1. Duality of the kinematically allowable displacements and statically allowable stresses resulting from the Lagrange-Castigliano principle /2, 3/ is known in linear elasticity theory. A corresponding assertion for surface displacements and stresses follows from dual variational principles for the boundary functionals in problems with bilateral and unilateral constraints on the boundary /4, 5/. The connectedness of the dual formulations of the variational problems (the explicit connection between the variables of the problems in terms of the governing relationships on the boundary) results in identical systems of boundary equations of the Ritz process.

As in /1/ we will give a brief description of the direct BEM formulation on the basis of a problem for a boundary functional

$$\min_{\varphi\in D} E(\varphi),\quad F(\varphi)=\int_S \varphi \mathbf{t}^{(\nu)}(\varphi)\,ds-2\int_S \varphi\mathbf{t}^{(\nu)}(\mathbf{u}^*)\,ds \tag{1.1}$$
$$D(\varphi)=\left\{\varphi \mid \mathbf{A}\varphi(x)=0,\quad x\in G,\quad \int_G \varphi\,dG=\int_G \operatorname{rot}\varphi\,dG=0\right\}$$

Here φ is the displacement vector, $\mathbf{A}$ is a vector operator of isotropic homogeneous elasticity theory, $G\subset E_m$ $(m=2,3)$ is a bounded domain with a sufficiently smooth boundary S with external normal $\boldsymbol{\nu}$, $\mathbf{t}^{(\nu)}(\mathbf{u}^*)$ is the vector of the given stresses at points of S.

**Prikl.Matem.Mekhan.*,55,1,118-125,1991

Realization of the solution of problem (1.1) is related /1/ to the approximation of the set of allowable functions D of the vector-potentials (DLP or SLP) with a desired density in the form of a complete interpolation polynomial. The polynomial coefficients are determined in terms of values of the displacement vector at BE nodes into which the boundary S is partitioned. The desired nodal values are parameters of BE-approximations "according to Ritz" /1/

$$\varphi_N = \sum_{i=1}^{m}\sum_{n=1}^{N}\sum_{k=1}^{K} \Phi_{nk}^{(i)}\beta_{nk}(x), \quad x \in G_\Delta \subset G \tag{1.2}$$
$$\int_{G_\Delta} \varphi_N \, dG_\Delta = \int_{G_\Delta} \operatorname{rot}\varphi_N \, dG_\Delta = 0$$

where $\Phi_{nk}^{(i)}$ $i = 1, \ldots, m)$, everywhere later) are components of the desired nodal BE displacements

$$\Delta s_n \subset S_\Delta = \bigcup_{n=1}^{N} \Delta s_n, \quad \operatorname{diam} \Delta s_n \to 0 \Rightarrow G_\Delta \to G$$

and β_{nk} are scalar "influence functions" of the k-th node, the n-th BE, constructed in the DLP /1/. The condition for the BE-approximation of the variational problem (1.1) to be solvable is presented in /1/. As a result of the Ritz process for solving problem (1.1) in the approximations (1.2), the system of BEM equations has the form /1/

$$\sum_{n=1}^{N}\sum_{k,\,p=1}^{K} \Phi_{nk}^{(i)} 2\mu \int_{\Delta s_n(\eta)} \partial_{\nu_n}\psi_k\psi_p \,|J|\, ds_n(\eta) = \sum_{n=1}^{N}\sum_{p=1}^{K} \int_{\Delta s_n(\eta)} t_i^{(\nu_n)}(\mathbf{u}_n{}^*)\,\psi_p \,|J|\, ds_n(\eta) \tag{1.3}$$

where $\psi_k(\eta)$, $\eta \in \Delta s_n$ are BEM basis functions corresponding to the selected interpolation polynomial /6/, $|J|$ is the determinant of the Jacobi matrix $[J]$ transforming the surface element $ds_n(\eta)'$ in the local coordinate system into a surface element $ds_n(y)$ in a global (Cartesian) coordinate system /6/.

System (1.3) is written for a special case when the vector-operator of the boundary stresses $\mathbf{t}^{(\nu)}(\varphi) = 2\mu\partial_\nu\varphi$ (for example, in problems of the torsion of an elastic isotropic homogeneous rod /7/) and its BE approximation in the approximations (1.2) has the form /1/

$$\mathbf{t}^{(\nu_n)}\Big(\sum_{i=1}^{m}\sum_{k=1}^{K} \Phi_{nk}^{(i)}\beta_{nk}\Big) = \sum_{i=1}^{m}\sum_{k=1}^{K} 2\mu\Phi_{nk}^{(i)}\partial_{\nu_n}\beta_{nk} \tag{1.4}$$

It has been shown /1/ that the BE-approximations (1.2) form a minimizing sequence for $F(\varphi)$ and converge as $N \to \infty$ to the generalized solution of the boundary-value problem with stresses $\mathbf{t}^{(\nu)}(\mathbf{u}^*)$ given on S, equivalent to the problem (1.1).

A formulation utilizing the SLP with density interpolating the stress field over nodal values of the displacements $\Phi_{nk}^{(i)}$ from (1.2) for the BE-approximations of the solution of the problem (1.1) can be an alternative formulation with respect to that elucidated. It is natural to expect that such a formulation will result in a system of BEM equations identical to (1.3).

Indeed, the displacement field at the points $\eta \in \Delta s_n$ is interpolated over the nodal values $\Phi_{nk}^{(i)}$ by a linear combination /1/

$$\varphi_n^{(i)}(\eta) = \sum_{k=1}^{K} \Phi_{nk}^{(i)}\psi_k(\eta)$$

while we take the BE-approximation of the "Ritz" solution of the problem (1.1) in the form

$$\varphi_N = \sum_{i=1}^{m}\sum_{n=1}^{N}\sum_{k=1}^{K} \Phi_{nk}^{(i)}\gamma_{nk}(x), \quad x \in G_\Delta$$
$$\int_{G_\Delta} \varphi_N dG_\Delta = \int_{G_\Delta} \operatorname{rot}\varphi_N \, dG_\Delta = 0$$

Here γ_{nk} are the "influence functions" of the k-th node, and the n-th BE are constructed according to the SLP that describe the displacement field at the points $x \in G_\Delta$

$$\gamma_n(x) = \frac{1}{2} \int_{\Delta s_n(y)} \Gamma^{(2)}(x, y)\, \mathbf{t}^{(\nu_n)} \Big(\sum_{i=1}^{m} \varphi_n^{(i)}(y) \Big)\, ds_n(y) \tag{1.5}$$

where $\Gamma^{(2)}(x, y)$ is Green's tensor of the second problem of statics /8/.

A linear combination of the vector-potentials (1.5) is a solution of the second problem of statics (in the domain G_Δ with boundary S_Δ)

$$\begin{gathered} \mathrm{A}\Big(\sum_{n=1}^{N} \gamma_n(x) \Big) = 0, \quad x \in G_\Delta \\ \mathbf{t}^{(\nu_\Delta)} \Big(\sum_{n=1}^{N} \gamma_n \Big) \Big|_{S_\Delta} = \sum_{n=1}^{N} \mathbf{t}^{(\nu_n)} \Big(\sum_{i=1}^{m} \varphi_n^{(i)}(y) \Big), \quad y \in \Delta s_n \end{gathered} \tag{1.6}$$

the BE-approximation of the boundary stress vector at the points $\eta \in \Delta s_n$ here has the form (1.4) (with β_{nk} replaced by γ_{nk}); then the "influence functions" γ_{nk} will be determined from the formula

$$\gamma_{nk}(x) = \frac{1}{2} \int_{\Delta s_n(\zeta)} \Gamma^{(2)}(x, y(\eta))\, 2\mu \partial_{\nu_n} \psi_k(\eta)\, |J|\, ds_n(\eta) \tag{1.7}$$

and are an SLP with scalar density. The boundary values of these potentials are the following:

$$\partial_{\nu_n} \gamma_{nk} |_{\Delta s_n} = \partial_{\nu_n} \psi_k(\eta) \Rightarrow \gamma_{nk} |_{\Delta s_n} = \psi_k(\eta), \quad \forall k = 1, \ldots, K; \quad \eta \in \Delta s_n \tag{1.8}$$

(this last equality holds apart from a certain constant, totally without influencing the derivation of the system of BEM equations).

Taking these boundary values into account, the Ritz system /1/, in which the "influence functions" β_{nk} are replaced by γ_{nk}, results in a system of BM equations of an identical system (1.3).

2. The dual BEM formulation for the solution of system (1.1) uses "Ritz" BE-approximations of the form

$$\begin{gathered} \varphi_N = \sum_{i=1}^{m} \sum_{n=1}^{N} \sum_{k=1}^{K} c^{-1}(\lambda, \mu)\, T_{nk}^{(i)} \gamma_{nk}(x), \quad x \in G_\Delta \\ \int_{G_\Delta} \varphi_N \, dG_\Delta = \int_{G_\Delta} \mathbf{rot}\, \varphi_N \, dG_\Delta = 0 \end{gathered} \tag{2.1}$$

where $T_{nk}^{(i)}$ are components of the desired nodal stresses, the BE $\Delta s_n \subset S_\Delta$, related to the components of the nodal displacements $\Phi_{nk}^{(i)}$ by physical relationships (for the case of the BE-approximation (1.4))

$$T_{nk}^{(i)} = c(\lambda, \mu)\, \Phi_{nk}^{(i)}, \; c = 2\mu, \; \forall k = 1, \ldots, K \tag{2.2}$$

and γ_{nk} are scalar "influence functions" of the form (1.7) constructed according to the SLP

$$\gamma_n(x) = \frac{1}{2} \int_{\Delta s_n(y)} \Gamma^{(2)}(x, y)\, \mathbf{t}^{(\nu_n)}(y)\, ds_n(y) \tag{2.3}$$

with the vector density

$$\mathbf{t}^{(\nu_n)}(y(\eta)) = \sum_{i=1}^{m} \sum_{k=1}^{K} T_{nk}^{(i)} \partial_{\nu_n} \psi_k(\eta), \quad \eta \in \Delta s_n \tag{2.4}$$

A linear combination of the vector-potentials γ_n is (like (1.6)) a solution of the second problem of statics (in the domain G_Δ with boundary S_Δ). Therefore, the approximations

$$\varphi_N \equiv \sum_{n=1}^{N} \gamma_n(x), \quad x \in G_\Delta$$

form a set $\{\varphi_N\}$, approximating the set of allowable vector-functions D of the variational problem (1.1). The boundary values (1.8) of the potentials γ_{nk} are used in changing from the Ritz system /1/ to the system of BEM equations. For the case of the BE approximation of the vector $\mathbf{t}^{(\nu)}(\varphi)$ of the form of (1.4), the system has the form

$$\sum_{n=1}^{N}\sum_{k,\,p=1}^{K} T_{nk}^{(i)} \int\limits_{\Delta s_n(\eta)} \partial_{\nu_n}\psi_k\psi_p\,|J|\,ds_n(\eta) = \sum_{n=1}^{N}\sum_{i=1}^{K} \int\limits_{\Delta s_n(\eta)} t_i^{(\nu_n)}(\mathbf{u}_n^*)\,\psi_p\,|J|\,ds_n(\eta) \tag{2.5}$$

identical to system (1.3) when (2.2) is taken into account, which confirms the deduction (see above) of the identity of the Ritz process systems for realizing solutions of related dual variational problems for boundary functionals; in the BEM formulation under consideration, the affinity mentioned is reflected by the governing (physical) relationships (2.2) between the stress and displacement components at the nodes.

The field of normal stresses at the points $\eta \in \Delta s_n \subset S_\Delta$ is determined (interpolated) from the nodal values $T_{nk}^{(i)}$ found using (2.4); the displacement field at the points $x \in G_\Delta$ is determined using (2.3). Therefore, the dual BEM formulation considered results, in the final analysis,in a variational solution in displacements of the original boundary-value problem; consequently, its foundation is analogous to the foundation for the direct formulation /1/.

3. We here utilize the BEM formulation on the basis of the variational problem for the generalized Trefftz functional /1/. The dual BEM formulation permitting direct approximation of the normal stress field at the BE points has an effective application for the realization of the solution of the BE-approximation of the variational problem for the generalized Trefftz functions corresponding to the elasticity theory problem for a piecewise-homogeneous medium /8/.

The solvability is established in the theory of boundary-value problems for elliptic equations (and systems) with discontinuous coefficients of the differential operator /9/ for such problems in an equivalent variational formulation in the Sobolev class of functions $W_2^1(G)$ (for second-order equations). A generalized Trefftz functional (in the example of the first problem) having the form

$$\Phi(\mathbf{u}) = 2\int\limits_G W_q(\mathbf{u})\,dG_q + \frac{1}{\alpha}\int\limits_{S_2}([\mathbf{t}^{(\nu_2)}(\mathbf{u})]_{S_2} - \alpha\mathbf{u})^2\,ds - \alpha\int\limits_{S_2}\mathbf{u}^2\,ds + \theta\int\limits_{S_1}\mathbf{u}^2\,ds, \quad G = G_1 \cup G_2; \quad \alpha, \theta = \mathrm{const} > 0 \tag{3.1}$$

is constructed /10/ for elasticity theory problems with discontinuous elasticity coefficients (here, unlike /10/, the norms of the boundary values $\mathbf{u}|_{S_1}$, $\mathbf{u}|_{S_2}$, $\mathbf{t}^{(\nu_2)}(\mathbf{u})|_{S_2}$) in L_2 are used for simplification).

Minimization of the functional $\Phi(u)$ by the allowable displacement vector-functions $\mathbf{u}$ satisfying the equation $\mathbf{A}_q\mathbf{u}(x) = \mathbf{K}$, $x \in G$ ($\mathbf{A}_q$ is the vector operator of isotropic elasticity theory with coefficients $\mu_q = (\mu_1, \mu_2)$, $\lambda_q = (\lambda_1, \lambda_2)$) results /10/ in a generalized solution $\mathbf{u}_0 \in W_2^{01}((G)$ ($W_2^{01}(G)$ is the subspace from $W_2^1(G)$ of vector-functions equal to zero on S_1) of the following boundary-value problem:

$$\begin{gathered}\mathbf{A}_q\mathbf{u}_0(x) = \mathbf{K}, \ x \in G \\ [\mathbf{u}_0]_{S_2} = 0, \quad [\mathbf{t}^{(\nu_2)}(\mathbf{u}_0)]_{S_2} = 0; \quad \mathbf{u}_0|_{S_1} = 0\end{gathered} \tag{3.2}$$

The solution $\mathbf{u}_0$ is understood in the sense of satisfying the integral identity

$$2\int\limits_G W_q(\mathbf{u}_0, \mathbf{v})\,dG_q - \int\limits_{S_2}[\mathbf{t}^{(\nu_2)}(\mathbf{u}_0)]_{S_2}\mathbf{v}\,ds - \int\limits_{S_1}\mathbf{t}^{(\nu_1)}(\mathbf{u}_0)\,\mathbf{v}\,ds = \int\limits_G \mathbf{K}\mathbf{v}\,dG_q, \quad \forall\mathbf{v} \in W_2^1(G) \tag{3.3}$$

The following notation is taken in (3.1)-(3.3): G is the domain occupied by a composite elastic medium, S_2 is the surface of discontinuity of the Lamé constants μ_q, λ_q; S_1 is the domain boundary $G_1 \supset G_2$, $S_1 \cap S_2 = \Phi$; $[\mathbf{u}]_{S_2} = \mathbf{u}_1 - \mathbf{u}_2$, u_1, u_2 are limit values of the vector $\mathbf{u}(x)$ as $x \to y \in S_2$ from the domains G_1 and G_2; $[\mathbf{t}^{(\nu_2)}(\mathbf{u})]_{S_2} = \mathbf{t}_1^{(\nu_2)}(\mathbf{u}_1) - \mathbf{t}_2^{(\nu_2)}(\mathbf{u}_2)$; it is assumed that the surfaces S_1 and S_2 are piecewise-continuous, $2W_q(u)$ ($q = 1, 2$ henceforth everywhere) is a quadratic form of the operator A_q.

Since the vector-function $\mathbf{u}_0 \in W_2^{01}(G) \cap W_2^2(G_q)$ is continuous at points of the domain

G /9/, the adjoint condition for the displacement vector $[\mathbf{u}_0]_{S_2} = 0$ is satisfied; the adjoint condition for the stress vector $[\mathbf{t}^{(\nu_2)}(\mathbf{u}_0)]_{S_2} = 0$ is satisfied for minimization of the functional $\Phi(\mathbf{u})$ /10/. The integral identity (3.3), obtained in /10/ by using the Betti formula is essentially separated into two identities that are satisfied $\forall \mathbf{v} \in W_2^1(G)$:

$$2\int_{G_1} W_1(\mathbf{u}_{01}, \mathbf{v})\, dG_1 + \int_{S_2} \mathbf{t}_1^{(\nu_2)}(\mathbf{u}_{01})\, \mathbf{v}\, ds - \int_{S_1} \mathbf{t}^{(\nu_1)}(\mathbf{u}_{01})\, \mathbf{v}\, ds = \int_{G_1} \mathbf{K}\mathbf{v}\, dG_1 \tag{3.4}$$

$$2\int_{G_2} W_2(\mathbf{u}_{02}, \mathbf{v})\, dG_2 - \int_{S_2} \mathbf{t}_2^{(\nu_2)}(\mathbf{u}_{02})\, \mathbf{v}\, ds = \int_{G_2} \mathbf{K}\mathbf{v}\, dG_2 \tag{3.5}$$

$$\mathbf{u}_0(x) = \begin{cases} \mathbf{u}_{01}(x), & x \in G_1 \\ \mathbf{u}_{02}(x), & x \in G_2 \end{cases} \tag{3.6}$$

Relationships (3.4)-(3.6) determine the set of allowable vector-functions of the problem of finding $\min \Phi(\mathbf{u})$. The BE-approximations of the solution of this problem is taken in the form of the superposition of potentials /1/: a volume and linear combination of SLP (see (2.1))

$$\mathbf{u}_{1N}(x) = \boldsymbol{\delta}_1(x) + \sum_{i=1}^{m} \sum_{n_1=1}^{N_1} \sum_{k=1}^{K} c_1^{-1} T_{n_1 k}^{(i)} \gamma_{n_1 k}(x) + \tag{3.7}$$

$$\sum_{i=1}^{m} \sum_{n_2=1}^{N_2} \sum_{k=1}^{K} c_1^{-1} T_{1n_2 k}^{(i)} \gamma_{1n_2 k}(x), \quad x \in G_{1\Delta}$$

$$\mathbf{u}_{2N}(x) = \boldsymbol{\delta}_2(x) + \sum_{i=1}^{m} \sum_{n_2=1}^{N_2} \sum_{k=1}^{K} c_2^{-1} T_{2n_2 k}^{(i)} \gamma_{2n_2 k}(x), \quad x \in G_{2\Delta}$$

$$c_q = c(\mu_q, \lambda_q), \quad \boldsymbol{\delta}_q(x) = \int_{G_{q\Delta}} \boldsymbol{\Gamma}_q(x, y)\, \mathbf{K}(y)\, dG_{q\Delta}(y)$$

$$S_{q\Delta} = \bigcup_{n_q=1}^{N_q} \Delta s_{n_q}$$

Here $S_{q\Delta}$ is the BE-approximation of the boundaries S_q, $T_{n_1 k}^{(i)}$, $T_{qn_2 k}^{(i)}$ are components of the desired nodal stresses, the influence functions $\gamma_{n_1 k}$, $\gamma_{qn_2 k}$ are defined by (1.7) (with 2μ replaced by appropriate constants c_1 and c_q); the matrices of the fundamental solutions $\boldsymbol{\Gamma}_q(x, y)$ and Green's tensors of the second problem of statics $\boldsymbol{\Gamma}_q^{(2)}(x, y)$ depend on μ_q, λ_q /8/.

The allowability of the application of (3.7) to solve the problem $\min \Phi(\mathbf{u})$ is established in detail in /1/ taking adjoint conditions into account in the form

$$U_{1n_2 k}^{(i)} = U_{2n_2 k}^{(k)}, \ \forall k = 1, \ldots, K, \ A n_2 = 1, \ldots, N_2$$

where $U_{qn_2 k}^{(i)}$ are components of the nodal displacements corresponding to components of the nodal stresses $T_{qn_2 k}^{(i)}$. Therefore, the displacement vector-functions are continuous at points of the surface of discontinuity of the Lamé constants. Satisfaction of the relationships (3.4) and (3.5) follows from the allowability of the approximations (3.7) (with integration over $G_{q\Delta}$ and $S_{q\Delta}$).

The Ritz process for solving the problem $\min \Phi(\mathbf{u})$ in the BE-approximations (3.7) results /1/ in a Ritz system. The procedure for eliminating volume integrals from the system is described in detail in /1, p.622/. Here the volume integrals of the form $2\int W_q(\mathbf{u}_{qN}, \mathbf{v})\, dG_{q\Delta}$ are eliminated by using relationships obtained from the integral identities (3.4) and (3.5) (with integration over $G_{q\Delta}$ and $S_{q\Delta}$).

The boundary values (1.8) of the SLP (of the form (1.7)) $\gamma_{n_1 k}$, $\gamma_{qn_2 k}$ are used in transferring to the system of BEM equations. As a result, the system of BEM equations to find the nodal stress components has the form

$$\sum_{n_2=1}^{N_2} \sum_{k,\, p=1}^{K} \Big[\alpha^{-1}(T_{1n_2 k}^{(i)} - T_{2n_2 k}^{(i)}) \int \partial_{\nu_{n_2}} \psi_k \partial_{\nu_{n_2}} \psi_p\, |J|\, ds - \tag{3.8}$$

$$(c_1^{-1} T_{1n_2 k}^{(i)} - c_2^{-1} T_{2n_2 k}^{(i)}) \int \psi_k \partial_{\nu_{n_2}} \psi_p\, |J|\, ds\Big] +$$

$$\sum_{n_1=1}^{N_1} \sum_{k,\, p=1}^{K} c_1^{-1} T_{n_1 k}^{(i)} \Big[\int \partial_{\nu_{n_1}} \psi_k \psi_p\, |J|\, ds + \theta c_1^{-1} \int \psi_k \psi_p\, |J|\, ds\Big] =$$

$$\sum_{n_2=1}^{N_2} \sum_{p=1}^{K} \Big[-\alpha^{-1} \int (t_i^{(\nu_{n_2})}(\boldsymbol{\delta}_1) - t_i^{(\nu_{n_2})}(\boldsymbol{\delta}_2))\, \partial_{\nu_{n_2}} \psi_p\, |J|\, ds +$$

$$
\left.\int (\delta_{1i} - \delta_{2i})\, \partial_{\mathbf{v}_{n_2}} \psi_p \,|J|\, ds\right] -
$$
$$
\sum_{n_1=1}^{N_1} \sum_{p=1}^{K} c_1^{-1} \left[\int t_i^{(\mathbf{v}_{n_1})} (\boldsymbol{\delta}_1)\, \psi_p \,|J|\, ds + \theta \int \delta_{1i} \psi_p \,|J|\, ds\right] -
$$
$$
\int_{G_{1\Delta}} K_i \gamma_{N_1K}\, dG_{1\Delta} - \int_{G_{2\Delta}} K_i \gamma_{2N_2K}\, dG_{2\Delta}
$$
$$
\gamma_{N_1K} = \sum_{n_1=1}^{N_1} \sum_{p=1}^{K} \gamma_{n_1p}, \quad \gamma_{2N_2K} = \sum_{n_2=1}^{N_1} \sum_{p=1}^{K} \gamma_{2n_2p}
$$

The integrals under the summation over n_q sign are evaluated according to the BE $\Delta s_{n_q}(\eta)$. The solution of system (3.8) is realized under the condition

$$
c_1^{-1} T_{1n_2k}^{(i)} = c_2^{-1} T_{2n_2k}^{(i)} \quad \forall k = 1, \ldots, K
$$

and the system is uniquely solvable /1/.

4. Let us examine certain questions of the numerical realization and practical utilization of the variational BEM formulas given above (see /1/ also). As an example, the Saint-Venant problem for a rod with elliptical transverse sections was considered, which can be formulated with respect to the scalar function of warping of a section as an inhomogeneous Neumann problem /7/ for the Laplace equation in the domain of the transverse section. This problem is equivalent to the problem of minimizing the boundary functional (of the form (1.1)) in a set of harmonic functions, or the problem of minimizing a generalized Trefftz functional (of the form (1.4), see /1/ also in the set mentioned.

The BE approximation of the solution of these variational problems by using harmonic double-layer potentials with desired density at the BE nodes results in systems of BE-equations (for which the mode of writing is presented in /1/) with symmetric matrices of coefficients of banded structure; the band width depends on the type of boundary elements utilized. The coefficients of the matrix of systems (of the form (1.3), (2.5), (3.8)) for the approximation of boundaries of isoparametric second-order BE /1/ are calculated most simply. A comparative analysis of the numerical results of the realization of the variational BEM formulations for such an approximation of the above-mentioned torsion problem, shows that to achieve accuracy of identical order for the "Trefftz" BE approximations, a number of BEs is required that is three times greater compared with the "Ritz" BE approximations; this confirms the known results /11/ about the slower convergence of the Trefftz method as compared with the Ritz method. It is also confirmed that the "Ritz" BE approximations are approximations "with excess" while "Trefftz" approximations are "with a disadvantage" as compared with the exact solution of the problem, this follows from a comparison of the corresponding nodal values of the warping function.

With respect to the practical application of the proposed BEM formulations, it should be noted that it is best to apply the formulation on the basis of minimizing the boundary functional in solving unilateral boundary-value problems. The "Ritz" BE approximations to solve the plane elasticity theory problem with unilateral constraints (of the generalized Signorini problem type) were used /12/ to realize the duality algorithm. It is convenient to apply the formulation on the basis of minimizing the generalized Trefftz functional when solving mixed problems and problems for piecewise-homogeneous elastic bodies since the singularities of these problems associated with different boundary conditions and adjoint conditions for the desired function are taken into account in the corresponding generalized Trefftz functions /10/.

The advangates and disadvantages of the proposed BEM formulation were analysed in detail /1/ compared with existing formulations on the basis of boundary integral equations /13, 6/ (which are applied more often in applications); consequently, without being repetitive, let us just emphasize that the development of a boundary ($(m-1)$-dimensional) BEM modification is started in /1/ and continued in this paper for the "Ritz" formulation with all the resultant features of the numerical realization of the algorithms inherent to the variational formulation of the BEM. If a comparison is made with the BEM formulation on the basis of the method of weighted residues (residuals) /14/, then it should be noted that the proposed formulations on the basis of variational problems for boundary functionals and generalized Trefftz functionals can be considered as a modification for realizing the method in question which also allows a variational formulation. A comparison of individual details indicates unconditionally the general features inherent in different BEM formulations: integral representations on the basis of potential theory and interpolation of the solution of the problem in the boundary element in a formulation using boundary integral equations or a variational approach.

We note that the proposed variational BEM formulations for solving boundary-value problems of linear elasticity theory can be treated as a semi-analytic BEM for solving elliptic boundary-value problems. The foundation for such terminology is the fact that the "Ritz" BE approximations of the solution of variational problems for boundary functionals (or generalized Trefftz functionals) with a constraint, (satisfaction of the differential equation of the boundary-value problem) equal identically the boundary potentials (or the superposition of volume and boundary potentials). Therefore, the constraint of the variational problems is satisfied exactly; the density of the boundary potentials is determined so that the boundary conditions of the boundary-value problem would be satisfied as a result of solving the variational problem for the boundary functional (or the generalized Trefftz functional).

REFERENCES

1. TERESHCHENKO V.YA., On certain formulations of the boundary-elements method. PMM, 51, 4, 1987.
2. RABOTNOV YU.N., Mechanics of a Deformable Solid Body. Nauka, Moscow, 1979.
3. GERMAIN P., Course in Mechanics of Continuous Media, Vysshaya Shkola, Moscow, 1983.
4. TERESHCHENKO V.YA., Dual variational problems for boundary functionals of linear elasticity theory, PMM, 44, 6, 1980.
5. TERESHCHENKO V.YA., On an approach to investigating the Signorini problem using the idea of duality, PMM, 46, 1, 1982.
6. BANERJI P. and BUTTERFIELD R., The Boundary-Elements Methods in Applied Sciences, Mir, Moscow, 1984.
7. TIMOSHENKO S.P. and GOODIER J., Theory of Elasticity, Nauka, Moscow, 1975.
8. KUPRADZE V.D., GEGELIA T.G., BASHELEISHVILI M.O., and BURCHULADZE T.V., Three-dimensional Problems of Mathematical Elasticity Theory, Izd. Tbilis. Univ. Tbilisi 1968.
9. LADYZHENSKAYA O.A. and URAL'TSEVA N.N., Linear and Quasilinear Equations of Elliptic Type, Nauka, Moscow, 1964.
10. TERESHCHENKO V.YA., Generalization of the Trefftz method for spatial problems of elasticity theory, Zh. Vychisl. Matem. Mat. Fiziki, 16, 4, 1976.
11. MIKHLIN S.G., Variational Methods in Mathematical Physics. Nauka, Moscow, 1970.
12. TERESHCHENKO V. YA., On an algorithm for the solution of the Signorini problem, PMM, 51, 6, 1987.
13. CRUZ T. and RIZZO F. (Eds)., The Method of Boundary Integral Equations, Mir, Moscow, 1978.
14. BREBBIA K. and WALKER S., Application of the Boundary-Elements Method in Engineering, Mir, Moscow, 1982.

Translated by M.D.F.